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GLOBAL BEHAVIOUR AND PERIODICITIES OF SOME FRACTIONAL RECURSIVE SEQUENCES

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Abstract. In this paper, we obtain the expression and form of the solutions of the following recursive sequences

$$x_{n+1} = \frac{x_{n-2}x_{n-7}}{x_{n-4}(\pm 1 \pm x_{n-2}x_{n-7})}, \quad n = 0, 1, ...,$$

where the initial conditions x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0 are arbitrary positive real numbers. We studied the equilibrium points of the given equation. Some qualitative properties such as the global stability, and the periodic character of the solutions in each case have been studied. We found expression and form of solution and presented some numerical examples by using random initial values in each case. Some figures have been given to explain the behavior of the obtained solutions by using MATLAB 6.5 to confirm the obtained results.

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1. Introduction

This paper deals with the solution behaviour of the recursive sequence

$$x_{n+1} = \frac{x_{n-2}x_{n-7}}{x_{n-4}(\pm 1 \pm x_{n-2}x_{n-7})}, \quad n = 0, 1, ..., \tag{1.1}$$

where the initial conditions x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0, are arbitrary positive real numbers. Also we obtain the form and study the solution of some special equations.

The theory and methods of discrete dynamical systems is varied field and developed greatly during the recent years, which impact almost every branch of mathematics. Every dynamical system $x_{n+1} = f(x_n, x_{n-2}, ..., x_{n-k})$ determines a difference equation and vise versa. Applications of difference equations also experienced gigantic improvement in many areas. One of the reasons for this is a necessity for some techniques whose can be used in investigating equations arising in mathematical models describing real life situations in population biology [36], economic, physics, probability theory, genetics, resource management, psychology, etc. There is no doubt that the theory of difference equations will continue to play an important role in mathematics as a whole. Nonlinear difference equations of order greater than one are of paramount importance in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, physiology, physics, engineering and economics. The study of properties of rational difference equations and systems of rational difference equations [1]-[18] has been an area of interest in recent years. The theory of difference equations occupies a central position in applicable analysis. Cinar [3] has obtained the solutions of the difference equation:.

$$x_{n+1} = \frac{ax_{n-1}}{1+bx_n x_{n-1}}.$$

Elabbasy et al. [8] studied the solutions of the rational recursive sequence

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=1}^{k} x_{n-i}}.$$

Karatas [16] gave the solution of the following difference equation

$$x_{n+1} = \frac{x_{n-5}}{1+x_{n-2}x_{n-5}}.$$

Simsek [46] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1+x_{n-1}}.$$

Yalçınkaya [51] has studied the boundedness, global stability, periodicity character and gave the solution of some special cases of the difference equation.

$$x_{n+1} = \frac{ax_{n-k}}{b+cx_n^p}.$$

See also [19]-[30]. Other related work on rational difference equations see in references [31]-[56].

Here, we recall some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let

$$F : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

A point $\overline{x} \in I$ is called an equilibrium point of Eq.(1.2) if

$$\overline{x} = F(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq.(1.2), or equivalently, $\overline{x}$ is a fixed point of F.

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Definition 1. *(Periodicity)*

A Sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Definition 2. *(Stability)*

(i) The equilibrium point $\overline{x}$ of Eq.(1.2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(1.2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(1.2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq.(1.2) is global attractor if for all

$$x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(1.2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(1.2).

(v) The equilibrium point $\overline{x}$ of Eq.(1.2) is unstable if $\overline{x}$ is not locally stable.

(vi) The linearized equation of Eq.(1.2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}.$$

Theorem A [44] Assume that $p, q \in R$ and $k \in \{0, 1, 2, ...\}$. Then

$$|p| + |q| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, ...$$

2. On the Equation $x_{n+1} = x_{n-2}x_{n-7}/x_{n-4}(1 + x_{n-2}x_{n-7})$

In this section, we give a specific form of the solution of the first equation in the form:

$$x_{n+1} = \frac{x_{n-2}x_{n-7}}{x_{n-4}(1 + x_{n-2}x_{n-7})}, \qquad n = 0, 1, ..., \tag{2.1}$$

where the initial values are arbitrary positive real numbers.

Theorem 2.1. *Let* $\{x_n\}_{n=-7}^{\infty}$ *be a solution of Eq.(2.1). Then, for* $n = 0, 1, \ldots$ *. Now, it follows that*

$$x_{30n-7} = h\prod_{i=0}^{n-1}\left(\frac{(1+(10i+3)bg)(1+(10i+6)af)(1+(10i)ch)}{(1+(10i+8)bg)(1+(10i+1)af)(1+(10i+5)ch)}\right)$$

$$x_{30n-6} = g\prod_{i=0}^{n-1}\left(\frac{(1+(10i)bg)(1+(10i+3)af)(1+(10i+7)ch)}{(1+(10i+5)bg)(1+(10i+8)af)(1+(10i+2)ch)}\right)$$

$$x_{30n-5} = f\prod_{i=0}^{n-1}\left(\frac{(1+(10i+7)bg)(1+(10i)af)(1+(10i+4)ch)}{(1+(10i+2)bg)(1+(10i+5)af)(1+(10i+9)ch)}\right)$$

$$x_{30n-4} = e\prod_{i=0}^{n-1}\left(\frac{(1+(10i+4)bg)(1+(10i+7)af)(1+(10i+1)ch)}{(1+(10i+9)bg)(1+(10i+2)af)(1+(10i+6)ch)}\right)$$

$$x_{30n-3} = d\prod_{i=0}^{n-1}\left(\frac{(1+(10i+1)bg)(1+(10i+4)af)(1+(10i+8)ch)}{(1+(10i+6)bg)(1+(10i+9)af)(1+(10i+3)ch)}\right)$$

$$x_{30n-2} = c\prod_{i=0}^{n-1}\left(\frac{(1+(10i+8)bg)(1+(10i+1)af)(1+(10i+5)ch)}{(1+(10i+3)bg)(1+(10i+6)af)(1+(10i+10)ch)}\right)$$

$$x_{30n-1} = b\prod_{i=0}^{n-1}\left(\frac{(1+(10i+5)bg)(1+(10i+8)af)(1+(10i+2)ch)}{(1+(10i+10)bg)(1+(10i+3)af)(1+(10i+7)ch)}\right)$$

$$x_{30n} = a\prod_{i=0}^{n-1}\left(\frac{(1+(10i+2)bg)(1+(10i+5)af)(1+(10i+9)ch)}{(1+(10i+7)bg)(1+(10i+10)af)(1+(10i+4)ch)}\right)$$

$$x_{30n+1} = \frac{ch}{e(1+ch)}\prod_{i=0}^{n-1}\left(\frac{(1+(10i+9)bg)(1+(10i+2)af)(1+(10i+6)ch)}{(1+(10i+4)bg)(1+(10i+7)af)(1+(10i+11)ch)}\right)$$

$$x_{30n+2} = \frac{bg}{d(1+bg)}\prod_{i=0}^{n-1}\left(\frac{(1+(10i+6)bg)(1+(10i+9)af)(1+(10i+3)ch)}{(1+(10i+11)bg)(1+(10i+4)af)(1+(10i+8)ch)}\right)$$

$$x_{30n+3} = \frac{af}{c(1+af)}\prod_{i=0}^{n-1}\left(\frac{(1+(10i+3)bg)(1+(10i+6)af)(1+(10i)ch)}{(1+(10i+8)bg)(1+(10i+11)af)(1+(10i+5)ch)}\right)$$

$$x_{30n+4} = \frac{ch}{b(1+2ch)}\prod_{i=0}^{n-1}\left(\frac{(1+(10i+10)bg)(1+(10i+3)af)(1+(10i+7)ch)}{(1+(10i+5)bg)(1+(10i+8)af)(1+(10i+12)ch)}\right)$$

$$x_{30n+5} = \frac{bg}{a(1+2bg)}\prod_{i=0}^{n-1}\left(\frac{(1+(10i+7)bg)(1+(10i+10)af)(1+(10i+4)ch)}{(1+(10i+12)bg)(1+(10i+5)af)(1+(10i+9)ch)}\right)$$

$$x_{30n+6} = \frac{eaf(1+ch)}{ch(1+2af)}\prod_{i=0}^{n-1}\left(\frac{(1+(10i+4)bg)(1+(10i+7)af)(1+(10i+11)ch)}{(1+(10i+9)bg)(1+(10i+12)af)(1+(10i+6)ch)}\right)$$

$$x_{30n+7} = \frac{dch(1+bg)}{bg(1+3ch)}\prod_{i=0}^{n-1}\left(\frac{(1+(10i+11)bg)(1+(10i+4)af)(1+(10i+8)ch)}{(1+(10i+6)bg)(1+(10i+9)af)(1+(10i+13)ch)}\right)$$

$$x_{30n+8} = \frac{cbg(1+af)}{af(1+3bg)}\prod_{i=0}^{n-1}\left(\frac{(1+(10i+8)bg)(1+(10i+11)af)(1+(10i+5)ch)}{(1+(10i+13)bg)(1+(10i+6)af)(1+(10i+10)ch)}\right)$$

$$x_{30n+9} = \frac{baf(1+2ch)}{ch(1+3af)}\prod_{i=0}^{n-1}\left(\frac{(1+(10i+5)bg)(1+(10i+8)af)(1+(10i+12)ch)}{(1+(10i+10)bg)(1+(10i+13)af)(1+(10i+7)ch)}\right)$$

$$x_{30n+10} = \frac{ach(1+2bg)}{bg(1+4ch)}\prod_{i=0}^{n-1}\left(\frac{(1+(10i+12)bg)(1+(10i+5)af)(1+(10i+9)ch)}{(1+(10i+7)bg)(1+(10i+10)af)(1+(10i+14)ch)}\right)$$

$$x_{30n+11} = \frac{bgch(1+2af)}{eaf(1+ch)(1+4bg)}\prod_{i=0}^{n-1}\left(\frac{(1+(10i+9)bg)(1+(10i+12)af)(1+(10i+6)ch)}{(1+(10i+14)bg)(1+(10i+7)af)(1+(10i+11)ch)}\right)$$

$$x_{30n+12} = \frac{afbg(1+3ch)}{dch(1+bg)(1+4af)}\prod_{i=0}^{n-1}\left(\frac{(1+(10i+6)bg)(1+(10i+9)af)(1+(10i+13)ch)}{(1+(10i+11)bg)(1+(10i+14)af)(1+(10i+8)ch)}\right)$$

$$x_{30n+13} = \frac{afh(1+3bg)}{bg(1+af)(1+5ch)}\prod_{i=0}^{n-1}\left(\frac{(1+(10i+13)bg)(1+(10i+6)af)(1+(10i+10)ch)}{(1+(10i+8)bg)(1+(10i+11)af)(1+(10i+15)ch)}\right)$$

$$x_{30n+14} = \frac{gch(1+3af)}{af(1+2ch)(1+5bg)}\prod_{i=0}^{n-1}\left(\frac{(1+(10i+10)bg)(1+(10i+13)af)(1+(10i+7)ch)}{(1+(10i+15)bg)(1+(10i+8)af)(1+(10i+12)ch)}\right)$$

$$x_{30n+15} = \frac{fbg(1+4ch)}{ch(1+2bg)(1+5af)}\prod_{i=0}^{n-1}\left(\frac{(1+(10i+7)bg)(1+(10i+10)af)(1+(10i+14)ch)}{(1+(10i+12)bg)(1+(10i+15)af)(1+(10i+9)ch)}\right)$$

$$x_{30n+16} = \frac{eaf(1+ch)(1+4bg)}{bg(1+2af)(1+6ch)}\prod_{i=0}^{n-1}\left(\frac{(1+(10i+14)bg)(1+(10i+7)af)(1+(10i+11)ch)}{(1+(10i+9)bg)(1+(10i+12)af)(1+(10i+16)ch)}\right)$$

$$x_{30n+17} = \frac{dch(1+bg)(1+4af)}{af(1+3ch)(1+6bg)}\prod_{i=0}^{n-1}\left(\frac{(1+(10i+11)bg)(1+(10i+14)af)(1+(10i+8)ch)}{(1+(10i+16)bg)(1+(10i+9)af)(1+(10i+13)ch)}\right)$$

$$x_{30n+17} = \frac{dch(1+bg)(1+4af)}{af(1+3ch)(1+6bg)}\prod_{i=0}^{n-1}\left(\frac{(1+(10i+11)bg)(1+(10i+14)af)(1+(10i+8)ch)}{(1+(10i+16)bg)(1+(10i+9)af)(1+(10i+13)ch)}\right)$$

$$x_{30n+18} = \frac{bg(1+af)(1+5ch)}{h(1+3bg)(1+6af)}\prod_{i=0}^{n-1}\left(\frac{(1+(10i+8)bg)(1+(10i+11)af)(1+(10i+15)ch)}{(1+(10i+13)bg)(1+(10i+16)af)(1+(10i+10)ch)}\right)$$

$$x_{30n+19} = \frac{af(1+2ch)(1+5bg)}{g(1+3af)(1+7ch)}\prod_{i=0}^{n-1}\left(\frac{(1+(10i+15)bg)(1+(10i+8)af)(1+(10i+12)ch)}{(1+(10i+10)bg)(1+(10i+13)af)(1+(10i+17)ch)}\right)$$

$$x_{30n+20} = \frac{ch(1+2bg)(1+5af)}{f(1+4ch)(1+7bg)}\prod_{i=0}^{n-1}\left(\frac{(1+(10i+12)bg)(1+(10i+15)af)(1+(10i+9)ch)}{(1+(10i+17)bg)(1+(10i+10)af)(1+(10i+14)ch)}\right)$$

$$x_{30n+21} = \frac{bg(1+2af)(1+6ch)}{e(1+ch)(1+4bg)(1+7af)}\prod_{i=0}^{n-1}\left(\frac{(1+(10i+9)bg)(1+(10i+12)af)(1+(10i+16)ch)}{(1+(10i+14)bg)(1+(10i+17)af)(1+(10i+11)ch)}\right)$$

$$x_{30n+22} = \frac{af(1+3ch)(1+6bg)}{d(1+bg)(1+4af)(1+8ch)}\prod_{i=0}^{n-1}\left(\frac{(1+(10i+16)bg)(1+(10i+9)af)(1+(10i+13)ch)}{(1+(10i+11)bg)(1+(10i+14)af)(1+(10i+18)ch)}\right)$$

where $x_{-7} = h,\ x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_0 = a.$

Proof. For $n = 0$, the result holds. Now, suppose that $n > 0$ and that our assumption holds for $n - 1$. That is,

$$x_{30n-37} = h\prod_{i=0}^{n-2}\left(\frac{(1+(10i+3)bg)(1+(10i+6)af)(1+(10i)ch)}{(1+(10i+8)bg)(1+(10i+1)af)(1+(10i+5)ch)}\right)$$

$$x_{30n-36} = g\prod_{i=0}^{n-2}\left(\frac{(1+(10i)bg)(1+(10i+3)af)(1+(10i+7)ch)}{(1+(10i+5)bg)(1+(10i+8)af)(1+(10i+2)ch)}\right)$$

$$x_{30n-35} = f\prod_{i=0}^{n-2}\left(\frac{(1+(10i+7)bg)(1+(10i)af)(1+(10i+4)ch)}{(1+(10i+2)bg)(1+(10i+5)af)(1+(10i+9)ch)}\right)$$

$$x_{30n-34} = e\prod_{i=0}^{n-2}\left(\frac{(1+(10i+4)bg)(1+(10i+7)af)(1+(10i+1)ch)}{(1+(10i+9)bg)(1+(10i+2)af)(1+(10i+6)ch)}\right)$$

$$x_{30n-33} = d\prod_{i=0}^{n-2}\left(\frac{(1+(10i+1)bg)(1+(10i+4)af)(1+(10i+8)ch)}{(1+(10i+6)bg)(1+(10i+9)af)(1+(10i+3)ch)}\right)$$

$$x_{30n-32} = c\prod_{i=0}^{n-2}\left(\frac{(1+(10i+8)bg)(1+(10i+1)af)(1+(10i+5)ch)}{(1+(10i+3)bg)(1+(10i+6)af)(1+(10i+10)ch)}\right)$$

$$x_{30n-31} = b\prod_{i=0}^{n-2}\left(\frac{(1+(10i+5)bg)(1+(10i+8)af)(1+(10i+2)ch)}{(1+(10i+10)bg)(1+(10i+3)af)(1+(10i+7)ch)}\right)$$

$$x_{30n-30} = a\prod_{i=0}^{n-2}\left(\frac{(1+(10i+2)bg)(1+(10i+5)af)(1+(10i+9)ch)}{(1+(10i+7)bg)(1+(10i+10)af)(1+(10i+4)ch)}\right)$$

$$x_{30n-29} = \frac{ch}{e(1+ch)}\prod_{i=0}^{n-2}\left(\frac{(1+(10i+9)bg)(1+(10i+2)af)(1+(10i+6)ch)}{(1+(10i+4)bg)(1+(10i+7)af)(1+(10i+11)ch)}\right)$$

$$x_{30n-28} = \frac{bg}{d(1+bg)}\prod_{i=0}^{n-2}\left(\frac{(1+(10i+6)bg)(1+(10i+9)af)(1+(10i+3)ch)}{(1+(10i+11)bg)(1+(10i+4)af)(1+(10i+8)ch)}\right)$$

$$x_{30n-27} = \frac{af}{c(1+af)}\prod_{i=0}^{n-2}\left(\frac{(1+(10i+3)bg)(1+(10i+6)af)(1+(10i)ch)}{(1+(10i+8)bg)(1+(10i+11)af)(1+(10i+5)ch)}\right)$$

$$x_{30n-26} = \frac{ch}{b(1+2ch)}\prod_{i=0}^{n-2}\left(\frac{(1+(10i+10)bg)(1+(10i+3)af)(1+(10i+7)ch)}{(1+(10i+5)bg)(1+(10i+8)af)(1+(10i+12)ch)}\right)$$

$$x_{30n-25} = \frac{bg}{a(1+2bg)}\prod_{i=0}^{n-2}\left(\frac{(1+(10i+7)bg)(1+(10i+10)af)(1+(10i+4)ch)}{(1+(10i+12)bg)(1+(10i+5)af)(1+(10i+9)ch)}\right)$$

$$x_{30n-24} = \frac{eaf(1+ch)}{ch(1+2af)}\prod_{i=0}^{n-2}\left(\frac{(1+(10i+4)bg)(1+(10i+7)af)(1+(10i+11)ch)}{(1+(10i+9)bg)(1+(10i+12)af)(1+(10i+6)ch)}\right)$$

$$x_{30n-23} = \frac{dch(1+bg)}{bg(1+3ch)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+11)bg)(1+(10i+4)af)(1+(10i+8)ch)}{(1+(10i+6)bg)(1+(10i+9)af)(1+(10i+13)ch)} \right)$$

$$x_{30n-22} = \frac{cbg(1+af)}{af(1+3bg)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+8)bg)(1+(10i+11)af)(1+(10i+5)ch)}{(1+(10i+13)bg)(1+(10i+6)af)(1+(10i+10)ch)} \right)$$

$$x_{30n-21} = \frac{baf(1+2ch)}{ch(1+3af)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+5)bg)(1+(10i+8)af)(1+(10i+12)ch)}{(1+(10i+10)bg)(1+(10i+13)af)(1+(10i+7)ch)} \right)$$

$$x_{30n-20} = \frac{ach(1+2bg)}{bg(1+4ch)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+12)bg)(1+(10i+5)af)(1+(10i+9)ch)}{(1+(10i+7)bg)(1+(10i+10)af)(1+(10i+14)ch)} \right)$$

$$x_{30n-19} = \frac{bgch(1+2af)}{eaf(1+ch)(1+4bg)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+9)bg)(1+(10i+12)af)(1+(10i+6)ch)}{(1+(10i+14)bg)(1+(10i+7)af)(1+(10i+11)ch)} \right)$$

$$x_{30n-18} = \frac{afbg(1+3ch)}{dch(1+bg)(1+4af)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+6)bg)(1+(10i+9)af)(1+(10i+13)ch)}{(1+(10i+11)bg)(1+(10i+14)af)(1+(10i+8)ch)} \right)$$

$$x_{30n-17} = \frac{afh(1+3bg)}{bg(1+af)(1+5ch)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+13)bg)(1+(10i+6)af)(1+(10i+10)ch)}{(1+(10i+8)bg)(1+(10i+11)af)(1+(10i+15)ch)} \right)$$

$$x_{30n-16} = \frac{gch(1+3af)}{af(1+2ch)(1+5bg)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+10)bg)(1+(10i+13)af)(1+(10i+7)ch)}{(1+(10i+15)bg)(1+(10i+8)af)(1+(10i+12)ch)} \right)$$

$$x_{30n-15} = \frac{fbg(1+4ch)}{ch(1+2bg)(1+5af)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+7)bg)(1+(10i+10)af)(1+(10i+14)ch)}{(1+(10i+12)bg)(1+(10i+15)af)(1+(10i+9)ch)} \right)$$

$$x_{30n-14} = \frac{eaf(1+ch)(1+4bg)}{bg(1+2af)(1+6ch)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+14)bg)(1+(10i+7)af)(1+(10i+11)ch)}{(1+(10i+9)bg)(1+(10i+12)af)(1+(10i+16)ch)} \right)$$

$$x_{30n-13} = \frac{dch(1+bg)(1+4af)}{af(1+3ch)(1+6bg)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+11)bg)(1+(10i+14)af)(1+(10i+8)ch)}{(1+(10i+16)bg)(1+(10i+9)af)(1+(10i+13)ch)} \right)$$

$$x_{30n-12} = \frac{bg(1+af)(1+5ch)}{h(1+3bg)(1+6af)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+8)bg)(1+(10i+11)af)(1+(10i+15)ch)}{(1+(10i+13)bg)(1+(10i+16)af)(1+(10i+10)ch)} \right)$$

$$x_{30n-11} = \frac{af(1+2ch)(1+5bg)}{g(1+3af)(1+7ch)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+15)bg)(1+(10i+8)af)(1+(10i+12)ch)}{(1+(10i+10)bg)(1+(10i+13)af)(1+(10i+17)ch)} \right)$$

$$x_{30n-10} = \frac{ch(1+2bg)(1+5af)}{f(1+4ch)(1+7bg)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+12)bg)(1+(10i+15)af)(1+(10i+9)ch)}{(1+(10i+17)bg)(1+(10i+10)af)(1+(10i+14)ch)} \right)$$

$$x_{30n-9} = \frac{bg(1+2af)(1+6ch)}{e(1+ch)(1+4bg)(1+7af)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+9)bg)(1+(10i+12)af)(1+(10i+16)ch)}{(1+(10i+14)bg)(1+(10i+17)af)(1+(10i+11)ch)} \right)$$

$$x_{30n-8} = \frac{af(1+3ch)(1+6bg)}{d(1+bg)(1+4af)(1+8ch)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+16)bg)(1+(10i+9)af)(1+(10i+13)ch)}{(1+(10i+11)bg)(1+(10i+14)af)(1+(10i+18)ch)} \right)$$

Now, it follows from Eq.(2.1) that,

$$x_{30n-6} = \frac{x_{30n-9}x_{30n-14}}{x_{30n-11}(1+x_{30n-9}x_{30n-14})}$$

$$= \frac{\frac{bg(1+2af)(1+6ch)}{e(1+ch)(1+4bg)(1+7af)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+9)bg)(1+(10i+12)af)(1+(10i+16)ch)}{(1+(10i+14)bg)(1+(10i+17)af)(1+(10i+11)ch)} \right) \frac{eaf(1+ch)(1+4bg)}{bg(1+2af)(1+6ch)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+14)bg)(1+(10i+7)af)(1+(10i+11)ch)}{(1+(10i+9)bg)(1+(10i+12)af)(1+(10i+16)ch)} \right)}{\frac{af(1+2ch)(1+5bg)}{g(1+3af)(1+7ch)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+15)bg)(1+(10i+8)af)(1+(10i+12)ch)}{(1+(10i+10)bg)(1+(10i+13)af)(1+(10i+17)ch)} \right) \left[1 + \frac{bg(1+2af)(1+6ch)}{e(1+ch)(1+4bg)(1+7af)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+9)bg)(1+(10i+12)af)(1+(10i+16)ch)}{(1+(10i+14)bg)(1+(10i+17)af)(1+(10i+11)ch)} \right) \frac{eaf(1+ch)(1+4bg)}{bg(1+2af)(1+6ch)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+14)bg)(1+(10i+7)af)(1+(10i+11)ch)}{(1+(10i+9)bg)(1+(10i+12)af)(1+(10i+16)ch)} \right) \right]}$$

$$= \frac{\frac{af}{(1+7af)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+7)af)}{(1+(10i+17)af)} \right)}{\frac{af(1+2ch)(1+5bg)}{g(1+3af)(1+7ch)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+15)bg)(1+(10i+8)af)(1+(10i+12)ch)}{(1+(10i+10)bg)(1+(10i+13)af)(1+(10i+17)ch)} \right) \left[1 + \frac{af}{(1+7af)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+7)af)}{(1+(10i+17)af)} \right) \right]}$$

$$= \frac{g(1+3af)(1+7ch)}{(1+2ch)(1+5bg)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+10)bg)(1+(10i+13)af)(1+(10i+17)ch)}{(1+(10i+15)bg)(1+(10i+8)af)(1+(10i+12)ch)} \right) \left[\frac{\frac{1}{(1+(10n-3)af)}}{\left[1+\frac{af}{(1+(10n-3)af)}\right]} \right]$$

$$= \frac{g(1+3af)(1+7ch)}{(1+2ch)(1+5bg)} \prod_{i=0}^{n-2} \left(\frac{(1+(10i+10)bg)(1+(10i+13)af)(1+(10i+17)ch)}{(1+(10i+15)bg)(1+(10i+8)af)(1+(10i+12)ch)} \right) \left[\frac{1}{(1+(10n-2)af)} \right]$$

$$x_{30n-6} = g \prod_{i=0}^{n-1} \left(\frac{(1+(10i)bg)(1+(10i+3)af)(1+(10i+7)ch)}{(1+(10i+5)bg)(1+(10i+8)af)(1+(10i+2)ch)} \right).$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed. □

Theorem 2.2. *Eq.(2.1) has a unique equilibrium point which is* $\overline{x} = 0$, *and is not locally asymptotically stable.*

Proof. We see that from Eq. (2.1)

$$\overline{x} = \frac{\overline{x}^2}{\overline{x}(1+\overline{x}^2)},$$

or

$$\overline{x}^2(1+\overline{x}^2-1) = 0,$$
$$\overline{x}^4 = 0.$$

Thus the equilibrium point of (2.1) is $\overline{x} = 0$.

Let $f : (0,\infty)^3 \longrightarrow (0,\infty)$ be a continuously differentiable function defined by

$$f(u,v,w) = \frac{vu}{w(1+vu)}.$$

Therefore at $\overline{x} = 0$

$$\left(\frac{\partial f}{\partial u}\right)_{\overline{x}} = 1, \quad \left(\frac{\partial f}{\partial v}\right)_{\overline{x}} = 1, \quad \left(\frac{\partial f}{\partial w}\right)_{\overline{x}} = -1.$$

The proof follows by using Theorem A □

For confirming the results of this section, we consider some numerical examples which represent different types of solutions to Eq.(2.1).

Example 1. *We assume* $x_{-7} = 13$, $x_{-6} = 7$, $x_{-5} = 19$, $x_{-4} = 10$, $x_{-3} = 15$, $x_{-2} = 8$, $x_{-1} = 10$, $x_0 = 12$. *(See Figure 1).*

Example 2. *See Figure 2 where* $x_{-7} = 9$, $x_{-6} = 15$, $x_{-5} = 11$, $x_{-4} = 10$, $x_{-3} = 7$, $x_{-2} = 8$, $x_{-1} = 10$, $x_0 = 4$.

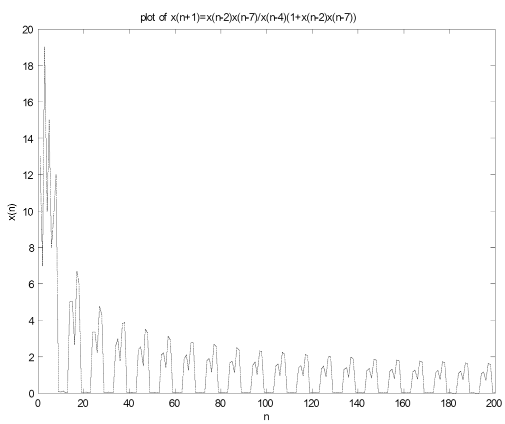

Figure 1.

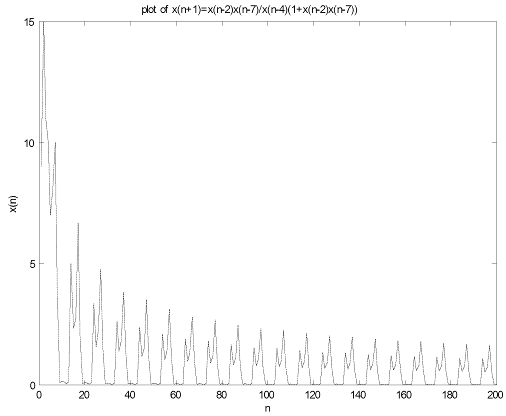

Figure 2.

3. On the Equation $x_{n+1} = x_{n-2}x_{n-7}/(x_{n-4}(-1 + x_{n-2}x_{n-7}))$

In this section, we give a specific form of the solution of the first equation in the form:

$$x_{n+1} = \frac{x_{n-2}x_{n-7}}{x_{n-4}(-1 + x_{n-2}x_{n-7})}, \qquad n = 0, 1, ..., \tag{3.1}$$

where the initial values are arbitrary nonzero real numbers.

Theorem 3.1. *Suppose $\{x_n\}_{n=-7}^{\infty}$ be a solution of Eq.(3.1). Then*

$$\begin{aligned}
x_{30n-7} &= \frac{h(-1+bg)^n}{(-1+af)^n(-1+ch)^n}, \quad x_{30n-6} = \frac{g(-1+af)^n(-1+ch)^n}{(-1+bg)^n},\\
x_{30n-5} &= \frac{f(-1+bg)^n}{(-1+af)^n(-1+ch)^n}, \quad x_{30n-4} = \frac{e(-1+af)^n(-1+ch)^n}{(-1+bg)^n},\\
x_{30n-3} &= \frac{d(-1+bg)^n}{(-1+af)^n(-1+ch)^n}, \quad x_{30n-2} = \frac{c(-1+af)^n(-1+ch)^n}{(-1+bg)^n},\\
x_{30n-1} &= \frac{b(-1+bg)^n}{(-1+af)^n(-1+ch)^n}, \quad x_{30n} = \frac{a(-1+af)^n(-1+ch)^n}{(-1+bg)^n},
\end{aligned}$$

$$\begin{aligned}
x_{30n+1} &= \frac{ch(-1+bg)^n}{e(-1+af)^n(-1+ch)^{n+1}}, \quad x_{30n+2} = \frac{bg(-1+af)^n(-1+ch)^n}{d(-1+bg)^{n+1}},\\
x_{30n+3} &= \frac{af(-1+bg)^n}{c(-1+af)^{n+1}(-1+ch)^n}, \quad x_{30n+4} = \frac{ch(-1+af)^n(-1+ch)^n}{b(-1+bg)^n},\\
x_{30n+5} &= \frac{bg(-1+bg)^n}{a(-1+af)^{n+1}(-1+ch)^n}, \quad x_{30n+6} = \frac{eaf(-1+ch)^{n+1}(-1+af)^n}{ch(-1+bg)^n},
\end{aligned}$$

$$\begin{aligned}
x_{30n+7} &= \frac{dch(-1+bg)^{n+1}}{bg(-1+ch)^{n+1}(-1+af)^n}, \quad x_{30n+8} = \frac{cbg(-1+af)^{n+1}(-1+ch)^n}{af(-1+bg)^{n+1}},\\
x_{30n+9} &= \frac{baf(-1+bg)^n}{ch(-1+ch)^n(-1+af)^{n+1}}, \quad x_{30n+10} = \frac{ach(-1+af)^n(-1+ch)^n}{bg(-1+bg)^n},\\
x_{30n+11} &= \frac{chbg(-1+bg)^n}{eaf(-1+ch)^{n+1}(-1+af)^n}, \quad x_{30n+12} = \frac{afbg(-1+ch)^{n+1}(-1+af)^n}{dch(-1+bg)^{n+1}},\\
x_{30n+13} &= \frac{afh(-1+bg)^{n+1}}{bg(-1+ch)^{n+1}(-1+af)^{n+1}}, \quad x_{30n+14} = \frac{gch(-1+ch)^n(-1+af)^{n+1}}{af(-1+bg)^{n+1}},
\end{aligned}$$

$$\begin{aligned}
x_{30n+15} &= \frac{fbg(-1+bg)^n}{ch(-1+ch)^n(-1+af)^{n+1}}, \quad x_{30n+16} = \frac{eaf(-1+ch)^{n+1}(-1+af)^n}{bg(-1+bg)^n},\\
x_{30n+17} &= \frac{dch(-1+bg)^{n+1}}{af(-1+ch)^{n+1}(-1+af)^n}, \quad x_{30n+18} = \frac{bg(-1+ch)^{n+1}(-1+af)^{n+1}}{h(-1+bg)^{n+1}},\\
x_{30n+19} &= \frac{af(-1+bg)^{n+1}}{g(-1+ch)^{n+1}(-1+af)^{n+1}}, \quad x_{30n+20} = \frac{ch(-1+ch)^n(-1+af)^{n+1}}{f(-1+bg)^{n+1}},\\
x_{30n+21} &= \frac{bg(-1+bg)^n}{e(-1+ch)^{n+1}(-1+af)^{n+1}}, \quad x_{30n+22} = \frac{af(-1+ch)^{n+1}(-1+af)^n}{d(-1+bg)^{n+1}},
\end{aligned}$$

where $x_{-7} = h,\ x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_0 = a$, and $af,\ ch,\ bg \neq 1$.

Proof. It follows from Eq.(3.1) that,

$$\begin{aligned}
x_{30n-7} &= \frac{x_{30n-10}x_{30n-15}}{x_{30n-12}(-1+x_{30n-10}x_{30n-15})}\\
&= \frac{\frac{ch(-1+ch)^{n-1}(-1+af)^n}{f(-1+bg)^n}\frac{fbg(-1+bg)^{n-1}}{ch(-1+ch)^{n-1}(-1+af)^n}}{\left[\frac{bg(-1+ch)^n(-1+af)^n}{h(-1+bg)^n}\right]\left[-1+\frac{ch(-1+ch)^{n-1}(-1+af)^n}{f(-1+bg)^n}\frac{fbg(-1+bg)^{n-1}}{ch(-1+ch)^{n-1}(-1+af)^n}\right]}\\
&= \frac{h(-1+bg)^n\frac{bg}{(-1+bg)}}{bg(-1+ch)^n(-1+af)^n\left[-1+\frac{bg}{(-1+bg)}\right]}\\
&= \frac{h(-1+bg)^n}{(-1+ch)^n(-1+af)^n\left[1-bg+bg\right]}\\
&= \frac{h(-1+bg)^n}{(-1+af)^n(-1+ch)^n}.
\end{aligned}$$

$$\begin{aligned}
x_{30n+1} &= \frac{x_{30n-2}x_{30n-7}}{x_{30n-4}(-1+x_{30n-2}x_{30n-7})}\\
&= \frac{\frac{c(-1+af)^n(-1+ch)^n}{(-1+bg)^n}\frac{h(-1+bg)^n}{(-1+af)^n(-1+ch)^n}}{\left[\frac{e(-1+af)^n(-1+ch)^n}{(-1+bg)^n}\right]\left[-1+\frac{c(-1+af)^n(-1+ch)^n}{(-1+bg)^n}\frac{h(-1+bg)^n}{(-1+af)^n(-1+ch)^n}\right]}\\
&= \frac{ch}{\left[\frac{e(-1+af)^n(-1+ch)^n}{(-1+bg)^n}\right][-1+ch]}\\
&= \frac{ch(-1+bg)^n}{e(-1+af)^n(-1+ch)^{n+1}}.
\end{aligned}$$

The other relations can be prove similarly. Thus the proof is completed. □

Theorem 3.2. *Eq.(3.1) has a periodic solutions with period ten iff* $af = bg = ch = 2$. *Moreover,* $\{x_n\}_{n=-7}^{\infty}$ *takes the form*

$$\left\{h, g, f, e, d, c, b, a, \frac{2}{e}, \frac{2}{d}, h, g, ...\right\}.$$

Proof. The proof follows from the forms of solutions of Eq. (3.1). □

Theorem 3.3. *Eq.(3.1) has two equilibrium points which are* $0, \pm\sqrt{2}$ *and these equilibrium points are not locally asymptotically stable.*

Proof. For the equilibrium points of (3.1), we can write,

$$\overline{x} = \frac{\overline{x}^2}{\overline{x}(-1+\overline{x}^2)},$$

or

$$\overline{x}^2(\overline{x}^2 - 2) = 0.$$

Thus, the equilibrium points of (3.1) are 0 and $\pm\sqrt{2}$.

Let $f : (0,\infty)^3 \longrightarrow (0,\infty)$ be a continuously differentiable function defined by

$$f(u,v,w) = \frac{vu}{w(-1+vu)}.$$

Therefore at $\overline{x} = 0$

$$\left(\frac{\partial f}{\partial u}\right)_{\overline{x}} = -1, \quad \left(\frac{\partial f}{\partial v}\right)_{\overline{x}} = -1, \quad \left(\frac{\partial f}{\partial w}\right)_{\overline{x}} = \pm 1.$$

The proof follows by using Theorem A □

Example 3. *We assume* $x_{-7} = 6,\ x_{-6} = 2,\ x_{-5} = 2.4,\ x_{-4} = 5,\ x_{-3} = 0.9,\ x_{-2} = 0.6,\ x_{-1} = 1.3,\ x_0 = 5.$ *(See Figure 3).*

Example 4. *See Figure 4 when we put* $x_{-7} = -4,\ x_{-6} = 3.2,\ x_{-5} = -9,\ x_{-4} = -6,\ x_{-3} = 0.9,\ x_{-2} = -0.5,\ x_{-1} = 5/8,\ x_0 = -2/9.$

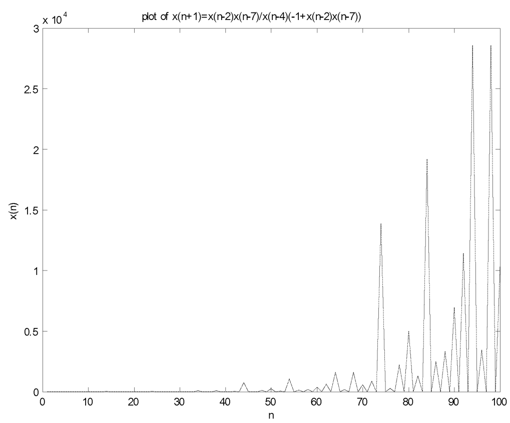

Figure 3.

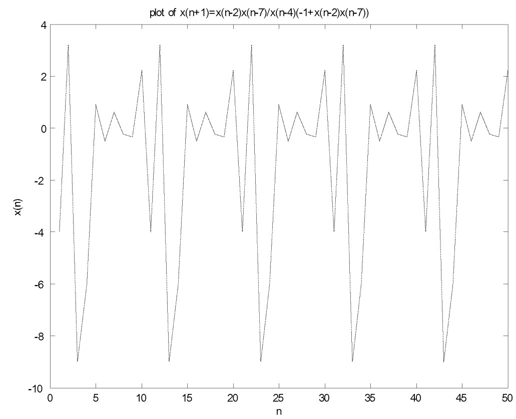

Figure 4.

The following theorem is similar to that of Theorem (2.1).

4. On the Equation $x_{n+1} = x_{n-2}x_{n-7}/(x_{n-4}(1 - x_{n-2}x_{n-7}))$

In this section, we give a specific form of the solution of the first equation in the form:

$$x_{n+1} = \frac{x_{n-2}x_{n-7}}{x_{n-4}(1 - x_{n-2}x_{n-7})}, \qquad n = 0, 1, ..., \tag{4.1}$$

Theorem 4.1. *Let* $\{x_n\}_{n=-7}^{\infty}$ *be a solution of (4.1). Then, for* $n = 0, 1, ...$

$$\begin{aligned}
x_{30n-7} &= h\prod_{i=0}^{n-1}\left(\frac{(1-(10i+3)bg)(1-(10i+6)af)(1-(10i)ch)}{(1-(10i+8)bg)(1-(10i+1)af)(1-(10i+5)ch)}\right) \\
x_{30n-6} &= g\prod_{i=0}^{n-1}\left(\frac{(1-(10i)bg)(1-(10i+3)af)(1-(10i+7)ch)}{(1-(10i+5)bg)(1-(10i+8)af)(1-(10i+2)ch)}\right) \\
x_{30n-5} &= f\prod_{i=0}^{n-1}\left(\frac{(1-(10i+7)bg)(1-(10i)af)(1-(10i+4)ch)}{(1-(10i+2)bg)(1-(10i+5)af)(1-(10i+9)ch)}\right) \\
x_{30n-4} &= e\prod_{i=0}^{n-1}\left(\frac{(1-(10i+4)bg)(1-(10i+7)af)(1-(10i+1)ch)}{(1-(10i+9)bg)(1-(10i+2)af)(1-(10i+6)ch)}\right)
\end{aligned}$$

$$
\begin{aligned}
x_{30n-3} &= d\prod_{i=0}^{n-1}\left(\frac{(1-(10i+1)bg)(1-(10i+4)af)(1-(10i+8)ch)}{(1-(10i+6)bg)(1-(10i+9)af)(1-(10i+3)ch)}\right)\\
x_{30n-2} &= c\prod_{i=0}^{n-1}\left(\frac{(1-(10i+8)bg)(1-(10i+1)af)(1-(10i+5)ch)}{(1-(10i+3)bg)(1-(10i+6)af)(1-(10i+10)ch)}\right)\\
x_{30n-1} &= b\prod_{i=0}^{n-1}\left(\frac{(1-(10i+5)bg)(1-(10i+8)af)(1-(10i+2)ch)}{(1-(10i+10)bg)(1-(10i+3)af)(1-(10i+7)ch)}\right)\\
x_{30n} &= a\prod_{i=0}^{n-1}\left(\frac{(1-(10i+2)bg)(1-(10i+5)af)(1-(10i+9)ch)}{(1-(10i+7)bg)(1-(10i+10)af)(1-(10i+4)ch)}\right)
\end{aligned}
$$

$$
\begin{aligned}
x_{30n+1} &= \frac{ch}{e(1-ch)}\prod_{i=0}^{n-1}\left(\frac{(1-(10i+9)bg)(1-(10i+2)af)(1-(10i+6)ch)}{(1-(10i+4)bg)(1-(10i+7)af)(1-(10i+11)ch)}\right)\\
x_{30n+2} &= \frac{bg}{d(1-bg)}\prod_{i=0}^{n-1}\left(\frac{(1-(10i+6)bg)(1-(10i+9)af)(1-(10i+3)ch)}{(1-(10i+11)bg)(1-(10i+4)af)(1-(10i+8)ch)}\right)\\
x_{30n+3} &= \frac{af}{c(1-af)}\prod_{i=0}^{n-1}\left(\frac{(1-(10i+3)bg)(1-(10i+6)af)(1-(10i)ch)}{(1-(10i+8)bg)(1-(10i+11)af)(1-(10i+5)ch)}\right)
\end{aligned}
$$

$$
\begin{aligned}
x_{30n+4} &= \frac{ch}{b(1-2ch)}\prod_{i=0}^{n-1}\left(\frac{(1-(10i+10)bg)(1-(10i+3)af)(1-(10i+7)ch)}{(1-(10i+5)bg)(1-(10i+8)af)(1-(10i+12)ch)}\right)\\
x_{30n+5} &= \frac{bg}{a(1-2bg)}\prod_{i=0}^{n-1}\left(\frac{(1-(10i+7)bg)(1-(10i+10)af)(1-(10i+4)ch)}{(1-(10i+12)bg)(1-(10i+5)af)(1-(10i+9)ch)}\right)\\
x_{30n+6} &= \frac{eaf(1-ch)}{ch(1-2af)}\prod_{i=0}^{n-1}\left(\frac{(1-(10i+4)bg)(1-(10i+7)af)(1-(10i+11)ch)}{(1-(10i+9)bg)(1-(10i+12)af)(1-(10i+6)ch)}\right)\\
x_{30n+7} &= \frac{dch(1-bg)}{bg(1-3ch)}\prod_{i=0}^{n-1}\left(\frac{(1-(10i+11)bg)(1-(10i+4)af)(1-(10i+8)ch)}{(1-(10i+6)bg)(1-(10i+9)af)(1-(10i+13)ch)}\right)
\end{aligned}
$$

$$
\begin{aligned}
x_{30n+8} &= \frac{cbg(1-af)}{af(1-3bg)}\prod_{i=0}^{n-1}\left(\frac{(1-(10i+8)bg)(1-(10i+11)af)(1-(10i+5)ch)}{(1-(10i+13)bg)(1-(10i+6)af)(1-(10i+10)ch)}\right)\\
x_{30n+9} &= \frac{baf(1-2ch)}{ch(1-3af)}\prod_{i=0}^{n-1}\left(\frac{(1-(10i+5)bg)(1-(10i+8)af)(1-(10i+12)ch)}{(1-(10i+10)bg)(1-(10i+13)af)(1-(10i+7)ch)}\right)\\
x_{30n+10} &= \frac{ach(1-2bg)}{bg(1-4ch)}\prod_{i=0}^{n-1}\left(\frac{(1-(10i+12)bg)(1-(10i+5)af)(1-(10i+9)ch)}{(1-(10i+7)bg)(1-(10i+10)af)(1-(10i+14)ch)}\right)\\
x_{30n+11} &= \frac{bgch(1-2af)}{eaf(1-ch)(1-4bg)}\prod_{i=0}^{n-1}\left(\frac{(1-(10i+9)bg)(1-(10i+12)af)(1-(10i+6)ch)}{(1-(10i+14)bg)(1-(10i+7)af)(1-(10i+11)ch)}\right)
\end{aligned}
$$

$$x_{30n+12} = \frac{afbg(1-3ch)}{dch(1-bg)(1-4af)} \prod_{i=0}^{n-1} \left(\frac{(1-(10i+6)bg)(1-(10i+9)af)(1-(10i+13)ch)}{(1-(10i+11)bg)(1-(10i+14)af)(1-(10i+8)ch)} \right)$$

$$x_{30n+13} = \frac{afh(1-3bg)}{bg(1-af)(1-5ch)} \prod_{i=0}^{n-1} \left(\frac{(1-(10i+13)bg)(1-(10i+6)af)(1-(10i+10)ch)}{(1-(10i+8)bg)(1-(10i+11)af)(1-(10i+15)ch)} \right)$$

$$x_{30n+14} = \frac{gch(1-3af)}{af(1-2ch)(1-5bg)} \prod_{i=0}^{n-1} \left(\frac{(1-(10i+10)bg)(1-(10i+13)af)(1-(10i+7)ch)}{(1-(10i+15)bg)(1-(10i+8)af)(1-(10i+12)ch)} \right)$$

$$x_{30n+15} = \frac{fbg(1-4ch)}{ch(1-2bg)(1-5af)} \prod_{i=0}^{n-1} \left(\frac{(1-(10i+7)bg)(1-(10i+10)af)(1-(10i+14)ch)}{(1-(10i+12)bg)(1-(10i+15)af)(1-(10i+9)ch)} \right)$$

$$x_{30n+16} = \frac{eaf(1-ch)(1-4bg)}{bg(1-2af)(1-6ch)} \prod_{i=0}^{n-1} \left(\frac{(1-(10i+14)bg)(1-(10i+7)af)(1-(10i+11)ch)}{(1-(10i+9)bg)(1-(10i+12)af)(1-(10i+16)ch)} \right)$$

$$x_{30n+17} = \frac{dch(1-bg)(1-4af)}{af(1-3ch)(1-6bg)} \prod_{i=0}^{n-1} \left(\frac{(1-(10i+11)bg)(1-(10i+14)af)(1-(10i+8)ch)}{(1-(10i+16)bg)(1-(10i+9)af)(1-(10i+13)ch)} \right)$$

$$x_{30n+18} = \frac{bg(1-af)(1-5ch)}{h(1-3bg)(1-6af)} \prod_{i=0}^{n-1} \left(\frac{(1-(10i+8)bg)(1-(10i+11)af)(1-(10i+15)ch)}{(1-(10i+13)bg)(1-(10i+16)af)(1-(10i+10)ch)} \right)$$

$$x_{30n+19} = \frac{af(1-2ch)(1-5bg)}{g(1-3af)(1-7ch)} \prod_{i=0}^{n-1} \left(\frac{(1-(10i+15)bg)(1-(10i+8)af)(1-(10i+12)ch)}{(1-(10i+10)bg)(1-(10i+13)af)(1-(10i+17)ch)} \right)$$

$$x_{30n+20} = \frac{ch(1-2bg)(1-5af)}{f(1-4ch)(1-7bg)} \prod_{i=0}^{n-1} \left(\frac{(1-(10i+12)bg)(1-(10i+15)af)(1-(10i+9)ch)}{(1-(10i+17)bg)(1-(10i+10)af)(1-(10i+14)ch)} \right)$$

$$x_{30n+21} = \frac{bg(1-2af)(1-6ch)}{e(1-ch)(1-4bg)(1-7af)} \prod_{i=0}^{n-1} \left(\frac{(1-(10i+9)bg)(1-(10i+12)af)(1-(10i+16)ch)}{(1-(10i+14)bg)(1-(10i+17)af)(1-(10i+11)ch)} \right)$$

$$x_{30n+22} = \frac{af(1-3ch)(1-6bg)}{d(1-bg)(1-4af)(1-8ch)} \prod_{i=0}^{n-1} \left(\frac{(1-(10i+16)bg)(1-(10i+9)af)(1-(10i+13)ch)}{(1-(10i+11)bg)(1-(10i+14)af)(1-(10i+18)ch)} \right)$$

where $x_{-7} = h,\ x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_0 = a.$

Theorem 4.2. *Eq.(4.1) has a unique equilibrium point which is* $\overline{x} = 0$, *and is not locally asymptotically stable.*

Example 5. *We assume* $x_{-7} = 6,\ x_{-6} = 1,\ x_{-5} = 9,\ x_{-4} = 5,\ x_{-3} = 7,\ x_{-2} = 1,\ x_{-1} = 3,\ x_0 = 4.$ *(See Figure 5).*

Example 6. *See Figure 6, when* $x_{-7} = 1.6,\ x_{-6} = 1.8,\ x_{-5} = .9,\ x_{-4} = 5,\ x_{-3} = .7,\ x_{-2} = 1.6,\ x_{-1} = 3,\ x_0 = .4.$

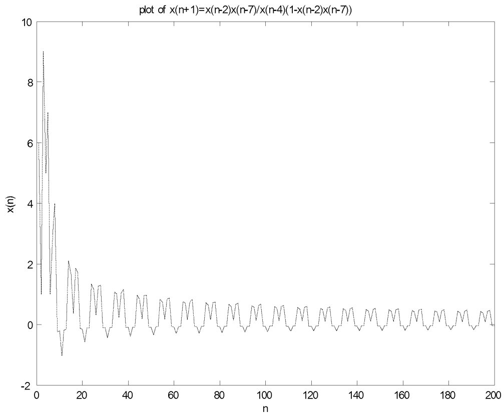

Figure 5.

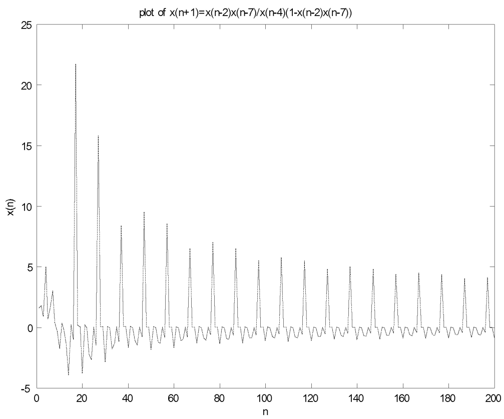

Figure 6.

The following theorem is similar to that of Theorem 3.1.

5. On the Equation $x_{n+1} = x_{n-2}x_{n-7}/(x_{n-4}(-1 - x_{n-2}x_{n-7}))$

In this section, we give a specific form of the solution of the first equation in the form:

$$x_{n+1} = \frac{x_{n-2}x_{n-7}}{x_{n-4}(-1 - x_{n-2}x_{n-7})}, \qquad n = 0, 1, ..., \tag{5.1}$$

where the initial values are arbitrary positive real numbers.

Theorem 5.1. *Assume that $\{x_n\}_{n=-7}^{\infty}$ be a solution of Eq.(5.1). Then*

$$\begin{aligned}
x_{30n-7} &= \frac{h(-1-bg)^n}{(-1-af)^n(-1-ch)^n}, \quad x_{30n-6} = \frac{g(-1-af)^n(-1-ch)^n}{(-1-bg)^n},\\
x_{30n-5} &= \frac{f(-1-bg)^n}{(-1-af)^n(-1-ch)^n}, \quad x_{30n-4} = \frac{e(-1-af)^n(-1-ch)^n}{(-1-bg)^n},\\
x_{30n-3} &= \frac{d(-1-bg)^n}{(-1-af)^n(-1-ch)^n}, \quad x_{30n-2} = \frac{c(-1-af)^n(-1-ch)^n}{(-1-bg)^n},\\
x_{30n-1} &= \frac{b(-1-bg)^n}{(-1-af)^n(-1-ch)^n}, \quad x_{30n} = \frac{a(-1-af)^n(-1-ch)^n}{(-1-bg)^n},
\end{aligned}$$

$$\begin{aligned}
x_{30n+1} &= \frac{ch(-1-bg)^n}{e(-1-af)^n(-1-ch)^{n+1}}, \quad x_{30n+2} = \frac{bg(-1-af)^n(-1-ch)^n}{d(-1-bg)^{n+1}},\\
x_{30n+3} &= \frac{af(-1-bg)^n}{c(-1-af)^{n+1}(-1-ch)^n}, \quad x_{30n+4} = \frac{ch(-1-af)^n(-1-ch)^n}{b(-1-bg)^n},\\
x_{30n+5} &= \frac{bg(-1-bg)^n}{a(-1-af)^{n+1}(-1-ch)^n}, \quad x_{30n+6} = \frac{eaf(-1-ch)^{n+1}(-1-af)^n}{ch(-1-bg)^n},
\end{aligned}$$

$$\begin{aligned}
x_{30n+7} &= \frac{dch(-1-bg)^{n+1}}{bg(-1-ch)^{n+1}(-1-af)^n}, \quad x_{30n+8} = \frac{cbg(-1-af)^{n+1}(-1-ch)^n}{af(-1-bg)^{n+1}},\\
x_{30n+9} &= \frac{baf(-1-bg)^n}{ch(-1-ch)^n(-1-af)^{n+1}}, \quad x_{30n+10} = \frac{ach(-1-af)^n(-1-ch)^n}{bg(-1-bg)^n},\\
x_{30n+11} &= \frac{chbg(-1-bg)^n}{eaf(-1-ch)^{n+1}(-1-af)^n}, \quad x_{30n+12} = \frac{afbg(-1-ch)^{n+1}(-1-af)^n}{dch(-1-bg)^{n+1}},\\
x_{30n+13} &= \frac{afh(-1-bg)^{n+1}}{bg(-1-ch)^{n+1}(-1-af)^{n+1}}, \quad x_{30n+14} = \frac{gch(-1-ch)^n(-1-af)^{n+1}}{af(-1-bg)^{n+1}},
\end{aligned}$$

$$\begin{aligned}
x_{30n+15} &= \frac{fbg(-1-bg)^n}{ch(-1-ch)^n(-1-af)^{n+1}}, \quad x_{30n+16} = \frac{eaf(-1-ch)^{n+1}(-1-af)^n}{bg(-1-bg)^n},\\
x_{30n+17} &= \frac{dch(-1-bg)^{n+1}}{af(-1-ch)^{n+1}(-1-af)^n}, \quad x_{30n+18} = \frac{bg(-1-ch)^{n+1}(-1-af)^{n+1}}{h(-1-bg)^{n+1}},\\
x_{30n+19} &= \frac{af(-1-bg)^{n+1}}{g(-1-ch)^{n+1}(-1-af)^{n+1}}, \quad x_{30n+20} = \frac{ch(-1-ch)^n(-1-af)^{n+1}}{f(-1-bg)^{n+1}},\\
x_{30n+21} &= \frac{bg(-1-bg)^n}{e(-1-ch)^{n+1}(-1-af)^{n+1}}, \quad x_{30n+22} = \frac{af(-1-ch)^{n+1}(-1-af)^n}{d(-1-bg)^{n+1}},
\end{aligned}$$

where $x_{-7} = h,\ x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_0 = a$, and $af,\ ch,\ bg \neq -1$.

Theorem 5.2. *Eq.(5.1) has a periodic solutions with period ten iff $af = bg = ch = 2$. Moreover, $\{x_n\}_{n=-7}^{\infty}$ takes the form*

$$\left\{h, g, f, e, d, c, b, a, \frac{-2}{e}, \frac{-2}{d}, h, g, ...\right\}.$$

Theorem 5.3. *Eq.(5.1) has a unique equilibrium point which is $\overline{x} = 0$ and this equilibrium point is not locally asymptotically stable.*

Example 7. *We assume $x_{-7} = 4.6,\ x_{-6} = 3.2,\ x_{-5} = 4,\ x_{-4} = -6,\ x_{-3} = .9,\ x_{-2} = 0.6,\ x_{-1} = 0.8,\ x_0 = 2.5$. (See Figure 7).*

Example 8. *Figure 8, shows solution when $x_{-7} = 4,\ x_{-6} = -3.2,\ x_{-5} = 9,\ x_{-4} = -6,\ x_{-3} = 0.9,\ x_{-2} = -0.5,\ x_{-1} = 5/8,\ x_0 = -2/9$.*

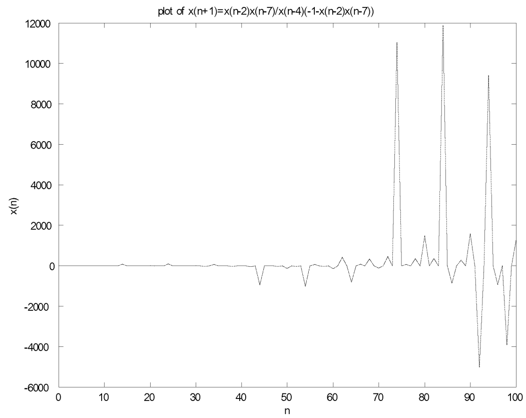

Figure 7.

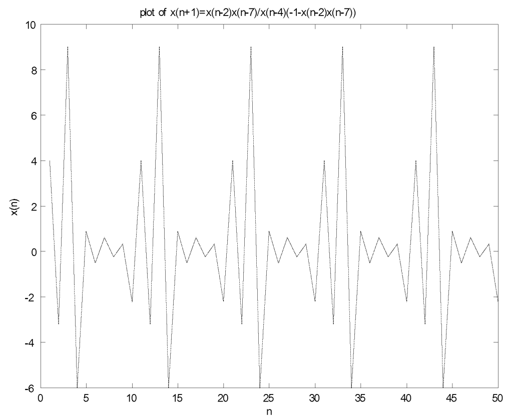

Figure 8.

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